

SHADOW PRICE IN THE POWER UTILITY CASE

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We consider the problem of maximizing expected power utility from consumption over an infinite horizon in the Black-Scholes model with proportional transaction costs, as studied in the paper [Shreve and Soner \(1994\)](#).

Similarly to [Kallsen and Muhle-Karbe \(2010\)](#), we derive a shadow price, that is, a frictionless price process with values in the bid-ask spread which leads to the same optimal policy. In doing so we explore and exploit the strong relationship between the shadow price and the Hamilton-Jacobi-Bellman-equation.

1. Introduction. It is a classical problem of mathematical finance to consider the problem of maximizing expected utility from consumption. This was initiated by [Merton \(1969, 1971\)](#), and thus is often referred to as the Merton problem. He found that for logarithmic or power utility it is optimal to keep a constant fraction of wealth in stocks and to consume at a rate proportional to current wealth.

This was extended to proportional transaction costs by [Magill and Constantinides \(1976\)](#). They stated that it is optimal to restrain from trading while the fraction of wealth invested in stocks is inside an interval $[\theta_1, \theta_2]$. Their heuristic argument was made precise by [Davis and Norman \(1990\)](#), which was then generalized by [Shreve and Soner \(1994\)](#) who managed to remove a couple of assumptions needed in [Davis and Norman \(1990\)](#).

These papers use methods from stochastic control. In recent years it seems there is more and more emphasis on solving portfolio optimization problems with transaction costs by determining the shadow price of the problem [Gerhold et al. \(2011\)](#); [Gerhold, Muhle-Karbe and Schachermayer \(2010\)](#); [Kallsen and Muhle-Karbe \(2010\)](#). This is a process that establishes a link between portfolio optimization with and without transaction costs as the optimal policy of the shadow price without frictions must coincide with that of the original problem.

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The first article in this context is [Kallsen and Muhle-Karbe \(2010\)](#). They use this dual approach to come up with a free boundary problem and solve that to derive the shadow price for logarithmic utility. They also showed a connection with the original solution of [Davis and Norman \(1990\)](#). They point out how the optimal consumption derived by Davis and Norman can be used to determine the shadow value process and from that the shadow price itself.

Our paper basically does the same for the power utility case. We start from the known solution of the problem with transaction costs and consider what that means for the shadow price. We notice that this optimal policy can only be derived from one shadow process, so it is quite straightforward to come up with a candidate. Then we only have to verify that this process satisfies the required properties of the shadow price.

The rest of the paper is organized as follows. Section 2 introduces the model and summarizes the known facts about problems with and without transaction costs for the power utility case. Section 3 shows how to come up with the candidate for the shadow price. Section 4 contains the verification, the main result is in Theorem [3.1](#).

2. Model and known results.

2.1. The model. We study the problem of maximizing expected utility from consumption over an infinite horizon in the presence of proportional transaction costs as in [Davis and Norman \(1990\)](#); [Shreve and Soner \(1994\)](#); [Kallsen and Muhle-Karbe \(2010\)](#). We consider a market with a bank account or bond whose value is constant 1 and a risky asset, a stock, whose price evolution is given by

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

with $S_0, \mu, \sigma > 0$, where W is a Brownian motion on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. Whenever trading occurs, the investor faces higher ask (buying) and lower bid (selling) prices, namely he can buy at $\bar{S}_t = (1 + \bar{\lambda})S_t$ and sell at $\underline{S}_t = (1 - \underline{\lambda})S_t$ for some $\bar{\lambda} \in (0, \infty)$ and $\underline{\lambda} \in (0, 1)$. As transactions of infinite variation lead to instantaneous bankruptcy, we limit ourselves to the following set of trading strategies.

DEFINITION 2.1. A trading strategy is a predictable $(\varphi_t^0, \varphi_t^1) \in \mathbb{R}^2$ process of finite variation, where φ_t^0 and φ_t^1 denote the number of units in the bond and the stock at time t respectively. A consumption rate is an adapted $c_t \in \mathbb{R}_+$ process, which satisfies $\int_0^t c_s ds < \infty$ a.s. for all $t \geq 0$. A pair $((\varphi_t^0, \varphi_t^1), c_t)$ of $(\varphi_t^0, \varphi_t^1)$ trading strategy and c_t consumption rate is called a portfolio/consumption process.

Writing $\varphi_t^1 = \varphi_t^\uparrow - \varphi_t^\downarrow$ as the difference between the cumulative number of shares bought (φ_t^\uparrow) and sold (φ_t^\downarrow) up to time t , a portfolio/consumption process is called self-financing, if

$$d\varphi_t^0 = -\overline{S}_t d\varphi_t^\uparrow + \underline{S}_t d\varphi_t^\downarrow - c_t dt$$

holds. This way a self-financing strategy can be determined by (φ_t^1, c_t) . From now on we only focus on self-financing strategies and use only the notation (φ_t^1, c_t) for them.

DEFINITION 2.2. A self-financing portfolio/consumption process is admissible if its liquidation value is non-negative, i.e.,

$$V_t^\varphi = \varphi_t^0 + \underline{S}_t \varphi_t^+ - \overline{S}_t \varphi_t^- \geq 0, \quad \text{a.s. for all } t \geq 0.$$

Given an initial endowment (x_0, y_0) , referring to the value of bonds and stocks respectively, an admissible portfolio/consumption process (φ_t^1, c_t) is optimal if it maximizes

$$\mathbf{E} \left(\int_0^\infty e^{-\delta t} u(c_t) dt \right)$$

over all admissible portfolio/consumption processes. Here $\delta > 0$ denotes a fixed given impatience rate, u a utility function, in our case the power utility, i.e., $u(c) = \frac{c^\gamma}{\gamma}$ for some $\gamma \in (0, 1)$.

The goal of this paper is to determine the shadow price process to this problem.

DEFINITION 2.3. A shadow price is a frictionless price process \tilde{S}_t , lying within the bid-ask spread ($\underline{S}_t \leq \tilde{S}_t \leq \overline{S}_t$ a.s.), such that the maximal expected utility for price process S_t with transaction costs and for price process \tilde{S}_t without transaction costs coincide.

Obviously, for any process lying in the bid-ask spread, the maximal expected utility is at least as high as for the original market with price process S_t , since the investor can trade at a smaller ask and a higher bid price. Indeed, this is what makes the shadow price so special, the optimal strategy with respect to it must only buy (resp. sell) when the shadow price coincides with the original ask (resp. bid) price.

2.2. *The problem without transaction costs.* In order to determine the shadow price, we need to obtain the optimal consumption and portfolio for the power utility when the stock price can be any Itô-process. In this section, we follow [Karatzas and Shreve \(1991\)](#) section 5.8.

Assume that the discounted price process \tilde{S} is an Itô process of the form

$$d\tilde{S}_t = \tilde{S}_t (\tilde{\mu}_t dt + \tilde{\sigma}_t dW_t),$$

where W is a Brownian motion on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. We also assume that there is a local martingale \tilde{Z} such that $\tilde{Z}\tilde{S}$ is a local martingale. This amounts to say that $\tilde{\mu}$ factorizes as $\tilde{\mu}_t = \tilde{r}_t \tilde{\sigma}_t$ with the “Sharpe-ratio” like process \tilde{r} , which is locally in L^2 almost surely. Then

$$\tilde{Z}_t = \exp \left\{ - \int_0^t \tilde{r}_s dW_s - \frac{1}{2} \int_0^t \tilde{r}_s^2 ds \right\}.$$

\tilde{Z} plays the role of the density process of an equivalent martingale measure. We use the present form to avoid technical difficulties arising from the possible nonexistence of an equivalent (local) martingale measure on \mathcal{F}_∞ .

The value process $\tilde{V} = \tilde{V}^{\varphi^1, c}$ of a self-financing strategy (φ^1, c) is

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \varphi_s^1 d\tilde{S}_s - \int_0^t c_s ds.$$

Note that if the strategy is admissible then $\tilde{V}_t \geq 0$ holds for all $t \geq 0$. Then for the discounted value process $\tilde{Z}_t \tilde{V}_t$

$$\begin{aligned} \tilde{Z}_t \tilde{V}_t - \tilde{V}_0 &= \int_0^t \varphi_s^1 \tilde{Z}_s d\tilde{S}_s - \int_0^t \tilde{Z}_s c_s ds + \int_0^t (\varphi_s^1 \tilde{S}_s + X_s) d\tilde{Z}_s + \int_0^t \varphi_s^1 d\langle \tilde{S}, \tilde{Z} \rangle_s = \\ &= \int_0^t \varphi_s^1 d(\tilde{Z}\tilde{S})_s - \int_0^t \tilde{Z}_s c_s ds + \int_0^t X_s d\tilde{Z}_s, \end{aligned}$$

where X_t denotes the balance of the bank account at time t . The point here is that $\tilde{Z}_t \tilde{V}_t + \int_0^t c_s \tilde{Z}_s ds$ is a local martingale. By stopping we get that

$$\tilde{V}_0 - \mathbf{E} \left(\tilde{Z}_{t \wedge \tau} \tilde{V}_{t \wedge \tau} \mid \mathcal{F}_0 \right) = \int_0^t \mathbf{E} \left(\tilde{Z}_s c_s \mathbf{1}_{(s \leq \tau)} \mid \mathcal{F}_0 \right) ds$$

Then letting t and τ go to infinity we get that

$$(1) \quad \tilde{V}_0 \geq \mathbf{E} \left(\int_0^\infty \tilde{Z}_s c_s ds \mid \mathcal{F}_0 \right),$$

provided that $c_t, \tilde{V}_t \geq 0$ for all $t \geq 0$. From this we can conclude that if a strategy is self-financing and admissible, then $\int_0^\infty \tilde{Z}_s c_s ds$ is finite almost surely and its generalized conditional expectation with respect to \mathcal{F}_0 exists and is finite almost surely. In the special case when $\mathbf{E}(\tilde{V}_0) < \infty$, which is usually assumed tacitly, we have that

$$\int_0^\infty \tilde{Z}_s c_s ds \in L^1.$$

From (1) an admissible strategy is optimal if

$$(2) \quad \tilde{V}_0 = \mathbf{E} \left(\int_0^\infty \tilde{Z}_s c_s ds \mid \mathcal{F}_0 \right)$$

and then

$$\tilde{Z}_t \tilde{V}_t + \int_0^t c_s \tilde{Z}_s ds = \mathbf{E} \left(\int_0^\infty \tilde{Z}_s c_s ds \mid \mathcal{F}_t \right)$$

is a true martingale. This also gives us

$$(3) \quad \tilde{Z}_t \tilde{V}_t = \mathbf{E} \left(\int_t^\infty \tilde{Z}_s c_s ds \mid \mathcal{F}_t \right),$$

that is to say that \tilde{V}_t is the present value of future consumption for the optimal policy, under the measure given by density process \tilde{Z} .

Observe also that if there exists an optimal strategy (φ^1, c) and the density process is \tilde{Z} then

$$(4) \quad \tilde{Z}_t \tilde{S}_t = \frac{1}{\varphi_t^1} \left(\mathbf{E} \left(\int_t^\infty \tilde{Z}_s c_s ds \mid \mathcal{F}_t \right) - \tilde{Z}_t X_t \right).$$

Now the optimal consumption is given by

$$c_t^* = I(\eta_t^*),$$

where $I = (u')^{-1}$ and

$$\eta_t^* = \eta_0^* \tilde{Z}_t e^{\delta t}.$$

For details, see [Karatzas and Shreve \(1991\)](#). The \mathcal{F}_0 measurable random variable η_0^* is determined by (2). For the power utility

$$u(x) = x^\gamma / \gamma, \quad u'(x) = x^{\gamma-1}, \quad I(x) = x^{1/(\gamma-1)}.$$

Hence η_0^* satisfies

$$\begin{aligned} \tilde{V}_0 = \mathbf{E} \left(\int_0^\infty \tilde{Z}_t (\eta_0^* e^{\delta t} \tilde{Z}_t)^{1/(\gamma-1)} dt \middle| \mathcal{F}_0 \right) = \\ (\eta_0^*)^{1/(\gamma-1)} \int_0^\infty e^{-\frac{1}{1-\gamma}\delta t} \mathbf{E} \left(\tilde{Z}_t^{\gamma/(\gamma-1)} \middle| \mathcal{F}_0 \right) dt. \end{aligned}$$

It is assumed that $\int_0^\infty e^{-\frac{1}{1-\gamma}\delta t} \mathbf{E} \left(\tilde{Z}_t^{\gamma/(\gamma-1)} \right) dt$ is finite and therefore η_0^* satisfying this equation exists. Then

$$c_t^* = c_0^* \tilde{Z}_t^{1/(\gamma-1)} e^{\delta t/(\gamma-1)},$$

where $c_0^* = (\eta_0^*)^{1/(\gamma-1)}$. Later, we need also that \tilde{Z} can be expressed using the optimal consumption rate

$$(5) \quad \tilde{Z}_t = e^{-\delta t} \frac{u'(c_t^*)}{u'(c_0^*)} = e^{-\delta t} \left(\frac{c_t^*}{c_0^*} \right)^{\gamma-1}.$$

2.3. The problem with transaction costs. In this section we summarize the results for the Merton problem in case of power utility with geometric Brownian motion price process. For details we refer to the papers [Davis and Norman \(1990\)](#); [Shreve and Soner \(1994\)](#) or the recent monograph [Kabanov and Safarian \(2009\)](#).

The endowment at time t is given by the pair (X_t, Y_t) , where X_t is the balance of the bank account and Y_t is the value of the stocks we hold evaluated using the mid-price S_t . Following [Shreve and Soner \(1994\)](#), we introduce the so-called Bellman or value function $v(x, y)$ which denotes the optimal value gained from consumption if we start with initial endowment (x, y) . Next, we denote by \mathcal{S} the solvency cone, which is the set of those (x, y) for which the liquidation value is still nonnegative:

$$\mathcal{S} = \{ (x, y) \in \mathbb{R}^2 : x + (1 + \bar{\lambda})y \geq 0 \text{ and } x + (1 - \underline{\lambda})y \geq 0 \}.$$

Then we have that the function v is C^2 in \mathcal{S} , except maybe the positive y -axis, and satisfies the following equation

$$(6) \quad \begin{aligned} \min \{ & \delta v(x, y) - \frac{1}{2} \sigma^2 y^2 v_{yy}(x, y) - \mu y v_y(x, y) - \tilde{u}(v_x(x, y)), \\ & - (1 - \underline{\lambda}) v_x(x, y) + v_y(x, y), (1 + \bar{\lambda}) v_x(x, y) - v_y(x, y) \} = 0, \end{aligned}$$

where $\tilde{u}(p) = \frac{1-\gamma}{\gamma} p^{-\frac{\gamma}{1-\gamma}}$ is the Legendre transform of $-u$.

This equation tells us the following. The second and the third parts come from the fact that for the optimal policy trading at a given situation, or more precisely trading differently than according to the optimal strategy cannot improve matters.

The first one expresses that the process

$$e^{-\delta t}v(X_t, Y_t) + \int_0^t e^{-\delta s}u(c_s)ds$$

must be a supermartingale and for the optimal policy a (local) martingale. This is not surprising if we consider that the first term above is the discounted value of the value process starting from endowment (X_t, Y_t) , and the second is the discounted value gained from the consumption process up to time t . So the first one is what can be gained from consumption after time t and the second is what we have gained so far following a strategy up to time t .

We can partition the solvency cone according to which one of the above equations is active, this way we get three sections, the no-trade (NT), the selling-stock (SS), and the buying-stock (BS) (or SMM) regions.

Our standing assumption is that the no-trade region (NT in the sequel) is contained in the first quadrant of \mathbb{R}^2 . We also suppose that the initial endowment (x_0, y_0) is in the NT.

In the no-trade region v satisfies the HJB-equation

$$(7) \quad \begin{aligned} \delta v(x, y) - \frac{1}{2}\sigma^2 y^2 v_{yy}(x, y) - \mu y v_y(x, y) - \tilde{u}(v_x(x, y)) &= 0, \\ (1 + \bar{\lambda})v_x(x, y) - v_y(x, y) &> 0, \\ -(1 - \underline{\lambda})v_x(x, y) + v_y(x, y) &> 0. \end{aligned}$$

The latter two come from the fact that it is not optimal to buy or sell stocks in the NT, see [Shreve and Soner \(1994\)](#). Note that they also mean that the fraction $\frac{v_y(x, y)}{v_x(x, y)}$ does not leave the interval $[1 - \underline{\lambda}, 1 + \bar{\lambda}]$. This will be important in the sequel.

For the SS section we have

$$(1 + \bar{\lambda})v_x(x, y) - v_y(x, y) = 0$$

and for the BS section

$$-(1 - \underline{\lambda})v_x(x, y) + v_y(x, y) = 0.$$

These equations correspond to the fact that when the fraction of wealth invested in stocks is too high (resp. too low), then it is optimal to sell (resp.

buy) stocks. Due to continuity, these hold true for the respective boundaries of the NT and are henceforth referred to as the boundary conditions of the HJB-equation.

We can deduce from this by differentiating with respect to x and y that for the SS section, except maybe for the positive y -axis,

$$(1 + \bar{\lambda})v_{xx}(x, y) - v_{xy}(x, y) = 0 \quad \text{and} \quad (1 + \bar{\lambda})v_{xy}(x, y) - v_{yy}(x, y) = 0$$

and for the BS section

$$-(1 - \underline{\lambda})v_{xx}(x, y) + v_{xy}(x, y) = 0 \quad \text{and} \quad -(1 - \underline{\lambda})v_{xy}(x, y) + v_{yy}(x, y) = 0.$$

Thus these equations also hold for the respective boundaries of the NT.

Using the homotheticity property of the Bellman function, for $y > 0$ we can write

$$v(x, y) = y^\gamma h\left(\frac{x}{y}\right)$$

with $h(u) = v(u, 1)$. It is easy to see that because of this connection, v and h have the same order of smoothness.

For this h function the HJB-equation can be written in the form

$$(8) \quad \left(\delta - \gamma\mu + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 \right) h(u) + (\mu - (1 - \gamma)\sigma^2) uh'(u) - \frac{1}{2}\sigma^2 u^2 h''(u) - \tilde{u}(h'(u)) = 0.$$

For later use, we introduce the notation

$$(9) \quad H(u) = \left(\delta - \gamma\mu + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 \right) h(u) + (\mu - (1 - \gamma)\sigma^2) uh'(u) - \frac{1}{2}\sigma^2 u^2 h''(u).$$

With this the HJB-equation becomes $H(u) = \tilde{u}(h'(u))$. By rearranging the above equation, we can express $h''(u)$ by the lower order terms, which means that h is smooth in $\left[\frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1} \right]$, where $[\theta_1, \theta_2]$ is the interval in which it is optimal to keep the fraction of wealth invested in stocks. Hence we get that v is smooth in the closure of the NT. Thus we can differentiate the HJB-equation freely with respect to x and y .

As for the boundary conditions of the HJB-equation, they take the following form for h

$$\begin{aligned}
 (-1 - u + \underline{\lambda})h'(u) + \gamma h(u) &> 0, \quad u \in \left[\frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1} \right), \\
 &= 0, \quad u = \frac{1-\theta_1}{\theta_1}, \\
 (1 + u + \bar{\lambda})h'(u) - \gamma h(u) &> 0, \quad u \in \left(\frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1} \right], \\
 &= 0, \quad u = \frac{1-\theta_2}{\theta_2}.
 \end{aligned}
 \tag{10}$$

Shreve and Soner, see [Shreve and Soner \(1994\)](#), have also shown that v is in C^2 except maybe the y -axis. So, these conditions should be supplemented with the so-called smooth pasting conditions which come from the aforementioned derivatives of the original boundary conditions

$$\begin{aligned}
 (-1 - u + \underline{\lambda})h''(u) - (1 - \gamma)h'(u) &= 0, \quad \text{for } u = \frac{1-\theta_1}{\theta_1}, \\
 (1 + u + \bar{\lambda})h''(u) + (1 - \gamma)h'(u) &= 0, \quad \text{for } u = \frac{1-\theta_2}{\theta_2}.
 \end{aligned}
 \tag{11}$$

Shreve and Soner have also proven that for the optimal policy we have

$$\begin{aligned}
 (X_t, Y_t) &\in \overline{NT}, \quad t \geq 0, \\
 dX_t &= -v_x(X_t, Y_t)^{-\frac{1}{1-\gamma}} dt + r_1(X_t, Y_t) dk_t, \\
 dY_t &= Y_t(\mu dt + \sigma dW_t) + r_2(X_t, Y_t) dk_t, \\
 dk_t &= \mathbb{1}_{((X_t, Y_t) \in \partial NT)} dk_t,
 \end{aligned}
 \tag{12}$$

where

$$r(x, y) = (r_1(x, y), r_2(x, y)) = \begin{cases} (-(1 + \bar{\lambda}), 1), & \text{if } (x, y) \in \partial NT \cap \partial BS, \\ (1 - \underline{\lambda}, -1), & \text{if } (x, y) \in \partial NT \cap \partial SS. \end{cases}$$

From [\(12\)](#) the consumption rate of the optimal policy is

$$c_t = v_x(X_t, Y_t)^{-\frac{1}{1-\gamma}}.
 \tag{13}$$

Note also, that r is such that for $(x, y) \in \partial NT$

$$v_x(x, y)r_1(x, y) + v_y(x, y)r_2(x, y) = 0,
 \tag{14}$$

and therefore for all $t \geq 0$

$$(v_x(X_t, Y_t)r_1(X_t, Y_t) + v_y(X_t, Y_t)r_2(X_t, Y_t))dk_t = 0.
 \tag{15}$$

Again we stress that these hold true if we use v_x or v_y instead of v .

3. Candidate for the shadow price process. Assume that there is a shadow price \tilde{S} . Then the optimal policy under the price S with transaction cost and under the price \tilde{S} without transaction cost have to be the same. Comparing identities (5) and (13) gives that the candidate for the martingale density is

$$\tilde{Z}_t = e^{-\delta t} \frac{v_x(X_t, Y_t)}{v_x(X_0, Y_0)},$$

where X, Y is the solution of (12).

Now, given (φ^1, c) and \tilde{Z} we can use (4) to get a candidate for the shadow price:

$$\tilde{Z}_t \tilde{S}_t = \frac{1}{\varphi_t^1} \left(\mathbf{E} \left(\int_t^\infty c_s \tilde{Z}_s ds \middle| \mathcal{F}_t \right) - X_t \tilde{Z}_t \right)$$

Observe that the special structure of the power utility function makes it possible to express the conditional expectation as

$$\mathbf{E} \left(\int_t^\infty c_s \tilde{Z}_s ds \middle| \mathcal{F}_t \right) = \mathbf{E} \left(\frac{1}{c_0^{\gamma-1}} \int_t^\infty e^{-\delta s} c_s^\gamma ds \middle| \mathcal{F}_t \right) = \frac{e^{-\delta t} \gamma v(X_t, Y_t)}{v_x(X_0, Y_0)}$$

and

$$X_t \tilde{Z}_t = X_t \frac{e^{-\delta t} v_x(X_t, Y_t)}{v_x(X_0, Y_0)}.$$

Since $v(x, y) = y^\gamma h\left(\frac{x}{y}\right)$ we have that

$$(16) \quad \gamma v(x, y) = x v_x(x, y) + y v_y(x, y).$$

Hence, using that $\varphi_t^1 = Y_t/S_t$,

$$\tilde{Z}_t \tilde{S}_t = \frac{S_t}{Y_t} \left(\frac{e^{-\delta t} Y_t v_y(X_t, Y_t)}{v_x(X_0, Y_0)} \right) = \frac{e^{-\delta t} S_t v_y(X_t, Y_t)}{v_x(X_0, Y_0)},$$

that is, the shadow price candidate \tilde{S} can be written as

$$(17) \quad \tilde{S}_t = \frac{S_t v_y(X_t, Y_t)}{v_x(X_t, Y_t)}.$$

THEOREM 3.1. *Assume that the no-trade region is inside the positive quadrant of \mathbb{R}^2 . Then \tilde{S} defined in (17) is the shadow price for the problem with transaction costs.*

PROOF. First, \tilde{S} lies in the bid-ask spread of the stock price S thanks to (6). Next, let us try to identify the local martingale density process. Define

$$Z_t = e^{-\delta t} \frac{v_x(X_t, Y_t)}{v_x(X_0, Y_0)}.$$

Then we have

$$\begin{aligned} \frac{dZ_t}{Z_t} &= -\delta dt + \frac{v_{xx}(X_t, Y_t)}{v_x(X_t, Y_t)} dX_t + \frac{v_{xy}(X_t, Y_t)}{v_x(X_t, Y_t)} dY_t + \frac{1}{2} \frac{v_{xyy}(X_t, Y_t)}{v_x(X_t, Y_t)} d\langle Y \rangle_t = \\ &= -\xi_t dW_t \end{aligned}$$

with $\xi_t = -\sigma Y_t \frac{v_{xy}(X_t, Y_t)}{v_x(X_t, Y_t)}$. The drift and the dk_t terms disappear due to the HJB-equation (7) and the boundary conditions (15) differentiated with respect to x .

Similar argument shows that

$$M_t = \frac{1}{v_x(X_0, Y_0)} e^{-\delta t} S_t v_y(X_t, Y_t)$$

is a local martingale. Indeed by Itô formula we get dynamics of M_t (we omit the arguments of the functions for brevity):

$$\begin{aligned} \frac{dM_t}{M_t} &= \left(-\delta + \mu + \sigma^2 Y_t \frac{v_{yy}}{v_y} \right) dt + \sigma dW_t + \frac{v_{xy}}{v_y} dX_t + \frac{v_{yy}}{v_y} dY_t + \frac{1}{2} \frac{v_{yyy}}{v_y} d\langle Y \rangle_t \\ &= \nu_t dW_t \end{aligned}$$

with $\nu_t = \sigma \left(\frac{Y_t v_{yy}(X_t, Y_t)}{v_y(X_t, Y_t)} + 1 \right)$. Here the drift and the dk_t term disappear due to the HJB-equation (7) and the boundary conditions (15) differentiated with respect to y .

Observe that $\tilde{S}_t = \frac{M_t}{Z_t}$, i.e., $\tilde{S}_t Z_t$ is a local martingale, and Z_t is indeed the local martingale measure density for the process \tilde{S}_t . Hence we get that the optimal consumption with respect to this process is

$$c_t^* = c_0^* Z_t^{1/(\gamma-1)} e^{\delta t/(\gamma-1)} = c_0^* \frac{v_x(X_t, Y_t)^{-1/(1-\gamma)}}{v_x(X_0, Y_0)^{-1/(1-\gamma)}}.$$

where c_0^* is determined by the identity

$$(18) \quad \tilde{V}_0 = c_0^* \int_0^\infty e^{-\frac{1}{1-\gamma} \delta t} \mathbf{E} \left(Z_t^{\gamma/(\gamma-1)} \mid \mathcal{F}_0 \right) dt$$

The left hand side of (18) is

$$\tilde{V}_0 = x_0 + \frac{y_0}{S_0} \tilde{S}_0 = \frac{xv_x(x_0, y_0) + y_0v_y(x_0, y_0)}{v_x(x_0, y_0)} = \gamma \frac{v(x_0, y_0)}{v_x(x_0, y_0)}.$$

In the last step we used (16). For the right hand side (18) we use that v is the value function and the optimal policy is described in terms of (X, Y) and v as in Subsection 2.3:

$$\begin{aligned} v(X_0, Y_0) &= \mathbf{E} \left(\int_0^\infty e^{-\delta t} u(c_t) dt \middle| \mathcal{F}_0 \right) = \\ &= \frac{1}{\gamma} \mathbf{E} \left(\int_0^\infty e^{-\delta t} v_x(X_t, Y_t)^{-\frac{\gamma}{1-\gamma}} dt \middle| \mathcal{F}_0 \right) \\ &= \frac{v_x^{-\gamma/(1-\gamma)}(X_0, Y_0)}{\gamma} \mathbf{E} \left(\int_0^\infty e^{-\frac{\delta}{1-\gamma} t} Z_t^{\gamma/(\gamma-1)} dt \middle| \mathcal{F}_0 \right). \end{aligned}$$

Then rearranging gives that $c_0^* = v_x(X_0, Y_0)^{-\frac{1}{1-\gamma}}$. Hence $c_t^* = v_x(X_t, Y_t)^{-\frac{1}{1-\gamma}}$, which is the coefficient of the drift term in the evolution of X_t .

Next we show that the optimal strategy (φ^1, c^*) of the problem with transaction cost is admissible and self-financing for the price process \tilde{S} without transaction cost.

The self-financing property follows from the fact that φ^1 only changes on the boundary of the no-trade region and due to the boundary conditions for v the shadow price candidate \tilde{S} and S coincide on the support of $d\varphi^1$.

The value process of this strategy is

$$\tilde{V}_t = \varphi_t^0 + \varphi_t^1 \tilde{S}_t = X_t + Y_t \frac{\tilde{S}_t}{S_t} \geq 0$$

as $(X_t, Y_t) \in \mathcal{S}$ and $\frac{\tilde{S}_t}{S_t} \in [1 - \underline{\lambda}, 1 + \bar{\lambda}]$. Thus this φ_t is an admissible strategy and c_t is the optimal consumption, we get that (φ_t, c_t) is the optimal portfolio/consumption pair for the process \tilde{S}_t and as the optimal strategy only trades on the boundaries, we have shown that \tilde{S}_t really is the shadow price. \square

REMARK 3.1. Note, that Z and M are true martingales. Hence Z is martingale density for \tilde{S} . The first part follows from the fact that ξ is uniformly bounded since

$$\xi_t = \sigma \left((1 - \gamma) + U_t \frac{h''(U_t)}{h'(U_t)} \right)$$

which is a continuous function of the bounded process $U_t = \frac{X_t}{Y_t}$. Here we use the assumption that the no-trade region does not meet the x -axis. This makes Z_t a true martingale.

For M we argue similarly.

$$\nu_t = \sigma \left(\frac{-\gamma(1-\gamma)h(U_t) + 2(1-\gamma)U_th'(U_t) + U_t^2h''(U_t)}{\gamma h(U_t) - U_th'(U_t)} + 1 \right)$$

is uniformly bounded, making M_t a true martingale.

4. Connection with work of Kallsen and Muhle-Karbe (2010).

For the shadow price, we need the function v . If we have such a function that satisfies the HJB-equation with the appropriate boundary conditions, then we can define the processes X and Y , and with that our candidate for the shadow price. Due to the homotheticity property, it is enough to determine the function h for which we have a free boundary problem, though it is not that simple to solve, see Davis and Norman (1990). In the following, we show how to transform it in order to come up with a somewhat simpler problem, which also demonstrates the connection of the HJB-equation and equation (3.15) in Kallsen and Muhle-Karbe (2010).

To derive the shadow price in the logarithmic utility case, Kallsen and Muhle-Karbe used the function g , which made the connection between C_t , the difference between the logarithm of the shadow and original price process, and β_t , the difference between the logarithm of the value held in stock and bond with respect to the shadow price, i.e. they showed that $C_t = g(\beta_t)$ and that this g function satisfies a free boundary problem.

In our case

$$C_t = \log \left(\frac{\tilde{S}_t}{S_t} \right) = \log \left(\frac{v_y(X_t, Y_t)}{v_x(X_t, Y_t)} \right)$$

and

$$(19) \quad \beta_t = \log \left(\frac{\varphi_t^1 \tilde{S}_t}{\varphi_t^0} \right) = \log \left(\frac{Y_t v_y(X_t, Y_t)}{X_t v_x(X_t, Y_t)} \right),$$

thus

$$(20) \quad C_t = \beta_t + \log \left(\frac{X_t}{Y_t} \right).$$

As $v(x, y) = y^\gamma h\left(\frac{x}{y}\right)$, we have that

$$\frac{v_y(x, y)}{v_x(x, y)} = \frac{\gamma h(u) - u h'(u)}{h'(u)},$$

with $u = \frac{x}{y}$. So g can be expressed as $g(z) = z + \log(f^{-1}(z))$, where

$$f(u) = \log \left(\frac{\gamma h(u) - u h'(u)}{u h'(u)} \right).$$

For this f we have

$$(21) \quad u f'(u) = -1 - (1 - \gamma) e^{-f(u)} - (1 + e^{-f(u)}) u \frac{h''(u)}{h'(u)}$$

and

$$(22) \quad u^2 f''(u) = -u f'(u) + u f'(u) (1 - \gamma) e^{-f(u)} + u f'(u) e^{-f(u)} u \frac{h''(u)}{h'(u)} \\ - (1 + e^{-f(u)}) \left(u \frac{h''(u)}{h'(u)} + u^2 \frac{h'''(u)}{h'(u)} - u^2 \frac{h''^2(u)}{h'^2(u)} \right).$$

From (21) we have

$$(23) \quad u \frac{h''(u)}{h'(u)} = -\frac{1 + (1 - \gamma) e^{-f(u)} - u f'(u)}{1 + e^{-f(u)}},$$

and as for (22), we need to express $u^2 \frac{h'''(u)}{h'(u)}$ in terms of $f(u)$. For this we turn to the HJB-equation for h and its derivative

$$(24) \quad \left(\delta + (1 - \gamma) \mu - \frac{1}{2} (1 - \gamma) (2 - \gamma) \sigma^2 \right) h'(u) + (\mu - (2 - \gamma) \sigma^2) u h''(u) \\ - \frac{1}{2} \sigma^2 u^2 h'''(u) + h'(u)^{-\frac{1}{1-\gamma}} h''(u) = 0.$$

With the notation (9) this becomes $H'(u) + h'(u)^{-\frac{1}{1-\gamma}} h''(u) = 0$.

Now from the HJB-equation, we can express $h'(u)^{-\frac{1}{1-\gamma}} = \frac{\gamma}{1-\gamma} \frac{\tilde{u}(h'(u))}{h'(u)}$ and putting this into (24), we get

$$(25) \quad u^2 \frac{h'''(u)}{h'(u)} = \left(\frac{2\delta}{\sigma^2} + (1 - \gamma) \frac{2\mu}{\sigma^2} - (1 - \gamma) (2 - \gamma) \right) \\ + \left(\frac{1}{1 - \gamma} \frac{2\delta}{\sigma^2} - \frac{\gamma}{1 - \gamma} \frac{2\mu}{\sigma^2} + \gamma \right) \frac{\gamma h(u)}{u h'(u)} u \frac{h''(u)}{h'(u)} \\ + \left(\frac{1}{1 - \gamma} \frac{2\mu}{\sigma^2} - 4 \right) u \frac{h''(u)}{h'(u)} - \frac{\gamma}{1 - \gamma} u^2 \frac{h''^2(u)}{h'^2(u)}.$$

Putting this and (23) into (22) and using $\frac{\gamma h(u)}{u h'(u)} = 1 + e^{f(u)}$, we have

$$\begin{aligned}
 & u^2 f''(u) + u f'(u) = \\
 & = \frac{1}{1-\gamma} \frac{2\delta}{\sigma^2} (1 + e^{f(u)}) - \frac{\gamma}{1-\gamma} \frac{2\mu}{\sigma^2} e^{f(u)} + \frac{\gamma^2}{1-\gamma} \frac{1}{1 + e^{-f(u)}} + \gamma e^{f(u)} \\
 (26) \quad & + \left(\frac{1}{1-\gamma} \frac{2\delta}{\sigma^2} (1 + e^{f(u)}) - \frac{1}{1-\gamma} \frac{2\mu}{\sigma^2} (\gamma(1 + e^{f(u)}) - 1) \right) u f'(u) \\
 & + \left(-1 + \frac{\gamma(3-\gamma)}{1-\gamma} \frac{1}{1 + e^{-f(u)}} + \gamma e^{f(u)} \right) u f'(u) \\
 & + \left(-1 + \frac{\gamma}{1-\gamma} \frac{1}{1 + e^{-f(u)}} + \frac{2}{1 + e^{-f(u)}} \right) u^2 f'^2(u).
 \end{aligned}$$

The boundary conditions (10) and (11) take the form

$$\begin{aligned}
 & \log u + f(u) > \log(1 - \underline{\lambda}), \quad u \in \left[\frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1} \right), \\
 & \quad \quad \quad = \log(1 - \underline{\lambda}), \quad u = \frac{1-\theta_1}{\theta_1}, \\
 (27) \quad & \log u + f(u) < \log(1 + \bar{\lambda}), \quad u \in \left(\frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1} \right], \\
 & \quad \quad \quad = \log(1 + \bar{\lambda}), \quad u = \frac{1-\theta_2}{\theta_2}, \\
 & u f'(u) = -1, \quad u = \frac{1-\theta_2}{\theta_2}, \frac{1-\theta_1}{\theta_1}.
 \end{aligned}$$

As $g(z) = z + \log(f^{-1}(z))$, with $z = f(u)$ we have

$$\begin{aligned}
 u f'(u) &= \frac{1}{g'(z) - 1} \\
 u^2 f''(u) + u f'(u) &= \frac{g''(z)}{(1 - g'(z))^3}.
 \end{aligned}$$

Thus (26), can be recast in terms of g yielding

$$(28) \quad g''(z) = a(z) + b(z)g'(z) + c(z)g'^2(z) + d(z)g'^3(z)$$

with coefficients

$$\begin{aligned}
 a(z) &= -\frac{2\mu}{\sigma^2} + 2(1-\gamma) \frac{1}{1 + e^{-z}}, \\
 b(z) &= -\frac{1}{1-\gamma} \frac{2\delta}{\sigma^2} (1 + e^z) + \frac{\gamma}{1-\gamma} \frac{2\mu}{\sigma^2} e^z + \frac{4\mu}{\sigma^2} - 1 + (5\gamma - 2) \frac{1}{1 + e^{-z}} - \gamma e^z, \\
 c(z) &= \frac{1}{1-\gamma} \frac{4\delta}{\sigma^2} (1 + e^z) - \frac{\gamma}{1-\gamma} \frac{4\mu}{\sigma^2} e^z - \frac{2\mu}{\sigma^2} + 1 + \frac{\gamma(4\gamma - 3)}{1-\gamma} \frac{1}{1 + e^{-z}} + 2\gamma e^z, \\
 d(z) &= -\frac{1}{1-\gamma} \frac{2\delta}{\sigma^2} (1 + e^z) + \frac{\gamma}{1-\gamma} \frac{2\mu}{\sigma^2} e^z - \frac{\gamma^2}{1-\gamma} \frac{1}{1 + e^{-z}} - \gamma e^z.
 \end{aligned}$$

Note that for $\gamma = 0$ this is exactly the same as (3.15) in [Kallsen and Muhle-Karbe \(2010\)](#).

As for the boundary conditions we have the following:

$$(29) \quad g(\underline{\beta}) = \log(1 + \bar{\lambda}), \quad g(\bar{\beta}) = \log(1 - \underline{\lambda}), \quad g'(\underline{\beta}) = 0, \quad g'(\bar{\beta}) = 0.$$

In order to construct the value function, we start with the solution of (28).

PROPOSITION 4.1. *Suppose that*

- (i) $(1 - \gamma)\sigma^2 > \mu > 0$ and
- (ii) $\delta - \gamma\mu + \frac{1}{2}\gamma(1 - \gamma)\sigma^2 > 0$.

Then the free boundary problem (28) with boundary conditions (29) has a strictly decreasing solution.

PROOF. The proof is practically the same as that of Proposition 4.2 in [Kallsen and Muhle-Karbe \(2010\)](#). We repeat the beginning of the proof as there is a misleading typo in it, the rest goes through with the obvious modifications, so we only sketch the main steps.

By assumption (i) there is a unique z_0 such that $a(z_0) = 0$. For any $\underline{\beta}_\Delta = z_0 - \Delta$ with $\Delta > 0$, there is a local solution g_Δ to (28) with initial conditions $g_\Delta(\underline{\beta}_\Delta) = \log(1 + \bar{\lambda})$ and $g'_\Delta(\underline{\beta}_\Delta) = 0$. Note that

$$\begin{aligned} b(z) &= d(z) + \frac{4\mu}{\sigma^2} - 1 + \frac{-4\gamma^2 + 7\gamma - 2}{1 - \gamma} \frac{1}{1 + e^{-z}} \\ c(z) &= -2d(z) - \frac{2\mu}{\sigma^2} + 1 + \frac{2\gamma^2 - 3\gamma}{1 - \gamma} \frac{1}{1 + e^{-z}}, \end{aligned}$$

and $\sup_{z \in \mathbb{R}} d(z) < 0$ by assumption (ii). Then for sufficiently large M' the sign of g'' is determined by $d(z)(g'(z))^3$ whenever $|g'(z)| > M'$. That is, we have $g''_\Delta(z) < 0$ for $g'_\Delta(z) > M'$ and $g''_\Delta(z) > 0$ for $g'_\Delta(z) < -M'$. Therefore g'_Δ can never leave the interval $[-M', M']$ for $z \geq \underline{\beta}_\Delta$.

Following [Kallsen and Muhle-Karbe \(2010\)](#) we let $\bar{\beta}_\Delta = \inf \left\{ z > \underline{\beta}_\Delta : g'(z) = 0 \right\}$. Since $a(\underline{\beta}_\Delta) < 0$, the function g_Δ is strictly decreasing on $[\underline{\beta}_\Delta, \bar{\beta}_\Delta]$.

Then it remains to show that for any $L < \log(1 + \bar{\lambda})$ there is a Δ such that $g_\Delta(\bar{\beta}_\Delta) = L$. The continuity argument of [Kallsen and Muhle-Karbe \(2010\)](#) applies to our case. First, it can be shown that $g_\Delta(\bar{\beta}_\Delta) \rightarrow \log(1 + \bar{\lambda})$ as $\Delta \rightarrow 0$. Next, it can be established that $\bar{\beta}_\Delta > z_0$. After that, it can be

seen that $g_\Delta(\bar{\beta}_\Delta) \rightarrow -\infty$ as $\Delta \rightarrow \infty$. Moreover, (g_Δ, g'_Δ) converges toward $(g_{\Delta_0}, g'_{\Delta_0})$ uniformly on compacts as $\Delta \rightarrow \Delta_0$. Finally, from the precedings it follows that $g_\Delta(\bar{\beta}_\Delta)$ depends continuously on Δ , which completes the proof. \square

Even though the proof works with $\delta - \gamma\mu + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0$, we are going to need a stricter version of it, which is a common standing assumption in related works

$$(30) \quad \delta > \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{\mu^2}{\sigma^2}.$$

This assumption is a necessary and sufficient condition for the value function in the problem without transaction costs to be finite and thus also a sufficient one when transaction costs are present. Observe that (30) implies $\delta - \gamma\mu + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0$.

Now that we have g , we can define

$$(31) \quad f(u) = \left(e^{g-id} \right)^{-1}(u)$$

Note that the domain of f is $[u_1, u_2]$, where

$$u_1 = \frac{1 - \underline{\lambda}}{e^{\underline{\beta}}}, \quad u_2 = \frac{1 + \bar{\lambda}}{e^{\underline{\beta}}}.$$

As g is strictly decreasing and $g: [\underline{\beta}, \bar{\beta}] \rightarrow [\log(1 - \underline{\lambda}), \log(1 + \bar{\lambda})]$, the function $f: [u_1, u_2] \rightarrow [\underline{\beta}, \bar{\beta}]$ is also strictly decreasing and satisfies the free boundary problem (26) with boundary conditions (27) by taking $\theta_1 = \frac{1}{1+u_2}$ and $\theta_2 = \frac{1}{1+u_1}$.

Next we define h as

$$(32) \quad h(u) = K e^{\gamma \int_{u_1}^u \frac{1}{v(1+e^{f(v)})} dv}$$

where K is chosen such that $H(u_1) = \tilde{u}(h'(u_1))$ is satisfied. For this $H(u_1) > 0$ is needed and as we are going to need it later on, we prove now that $H(u)$ is always positive.

From the definition of h , we have

$$\begin{aligned} \frac{h'(u)}{h(u)} &= \frac{\gamma}{u(1+e^{f(u)})} \\ \frac{h''(u)}{h(u)} &= \frac{\gamma}{u^2(1+e^{f(u)})^2} \left(\gamma - 1 - (1+uf'(u))e^{f(u)} \right). \end{aligned}$$

Using that $uf'(u) \geq -1$, we get

$$\begin{aligned}
\frac{H(u)}{h(u)} &= \left(\delta - \gamma\mu + \frac{1}{2}\gamma(1-\gamma)\sigma^2 \right) + (\mu - (1-\gamma)\sigma^2) \frac{\gamma}{1+e^{f(u)}} \\
&\quad - \frac{1}{2}\sigma^2 \frac{\gamma}{(1+e^{f(u)})^2} \left(\gamma - 1 - (1+uf'(u))e^{f(u)} \right) \\
&\geq \left(\delta - \gamma\mu + \frac{1}{2}\gamma(1-\gamma)\sigma^2 \right) + \frac{\gamma(\mu - (1-\gamma)\sigma^2)}{1+e^{f(u)}} + \frac{1}{2}\sigma^2 \frac{\gamma(1-\gamma)}{(1+e^{f(u)})^2} \\
&= \delta - \gamma\mu \frac{1}{1+e^{-f(u)}} + \frac{1}{2}\gamma(1-\gamma)\sigma^2 \frac{1}{(1+e^{-f(u)})^2}
\end{aligned}$$

with equality at $u = u_1, u_2$. This is a second-order polynomial of $\frac{1}{1+e^{-f(u)}}$. As the minimum is attained at $\frac{\mu}{(1-\gamma)\sigma^2} \in (\frac{1}{1+e^{-\underline{\beta}}}, \frac{1}{1+e^{-\bar{\beta}}})$, we need that this polynomial is always positive, which holds if and only if

$$\mu^2\gamma^2 < 2\delta\gamma(1-\gamma)\sigma^2 \Leftrightarrow \frac{1}{2} \frac{\gamma}{1-\gamma} \frac{\mu^2}{\sigma^2} < \delta.$$

This is why we need this assumption.

The fact that f satisfies (26), yields that the function h satisfies (25). From this it can be shown that h satisfies the HJB-equation. Let us start with the boundary conditions. The definition of h implies that

$$(33) \quad f(u) = \log \left(\frac{\gamma h(u)}{uh'(u)} - 1 \right)$$

meaning that

$$\frac{\gamma h(u)}{uh'(u)} = 1 + e^{f(u)}$$

which is again strictly decreasing, giving that

$$\begin{aligned}
\frac{\gamma h(u_1)}{u_1 h'(u_1)} &= 1 + e^{\underline{\beta}}, \\
1 + e^{\underline{\beta}} &< \frac{\gamma h(u)}{uh'(u)} < 1 + e^{\bar{\beta}}, \quad u \in (u_1, u_2), \\
\frac{\gamma h(u_2)}{u_2 h'(u_2)} &= 1 + e^{\bar{\beta}}.
\end{aligned}$$

Using the definitions of u_1 and u_2 , it is clear that (10) holds as $uh'(u)$ is always positive. To get the smooth pasting conditions, we take a look

at the boundary conditions of f , namely $uf'(u) = -1$ for $u = u_1, u_2$. By differentiating (33), we get

$$u \frac{h''(u)}{h'(u)} = -(1 - \gamma) \frac{1}{1 + e^{f(u)}} - \frac{1 + uf'(u)}{1 + e^{-f(u)}}$$

meaning that

$$\begin{aligned} u_1 \frac{h''(u_1)}{h'(u_1)} &= -(1 - \gamma) \frac{u_1}{1 + u_1 - \underline{\lambda}} \\ u_2 \frac{h''(u_2)}{h'(u_2)} &= -(1 - \gamma) \frac{u_2}{1 + u_2 + \bar{\lambda}}. \end{aligned}$$

Thus (11) holds as well.

The only thing left to be shown is that h satisfies the equation (8). This follows from (25). If we multiply it by $\frac{1}{2}\sigma^2 h'(u)$ and rearrange it, we have

$$-H'(u) = \frac{\gamma h''(u)}{(1 - \gamma)h'(u)} H(u),$$

which we can divide by H to get

$$\frac{H'(u)}{H(u)} = -\frac{\gamma}{1 - \gamma} \frac{h''(u)}{h'(u)}.$$

This yields $K_1 H(u) = \tilde{u}(h'(u))$ with some constant K_1 . Since $H(u_1) = \tilde{u}(h'(u_1))$ by the definition of h we have that $K_1 = 1$. Hence the HJB-equation (8) holds for h , which in turn means that $v(x, y) = y^\gamma h\left(\frac{x}{y}\right)$ satisfies the HJB-equation (7) on $\{(x, y) \in \mathbb{R}^2 : y > 0, u_1 < x/y < u_2\}$.

To summarize, we obtained the following result

PROPOSITION 4.2. *Assume that (30) and $(1 - \gamma)\sigma^2 > \mu > 0$ hold. Let g be the solution to the free boundary problem (28), (29). Define f, h as in (31), (32), respectively and let*

$$v(x, y) = y^\gamma h\left(\frac{x}{y}\right), \quad \text{for } (x, y) \in NT,$$

where $NT = \{(x, y) \in \mathbb{R}^2 : y > 0, u_1 < \frac{x}{y} < u_2\}$. The function v and its derivatives extend continuously to \overline{NT} .

Then v is a solution to (7) on NT , satisfies

$$(34) \quad (1 - \underline{\lambda})v_x(x, y) \leq v_y(x, y) \leq (1 + \bar{\lambda})v_x(x, y), \quad \text{for } (x, y) \in \overline{NT}$$

and the boundary conditions

$$\begin{aligned} (1 + \bar{\lambda})v_x(u_1y, y) &= v_y(u_1y, y), & (1 - \underline{\lambda})v_x(u_2y, y) &= v_y(u_2y, y), \\ (1 + \bar{\lambda})v_{xx}(u_1y, y) &= v_{yx}(u_1y, y), & (1 - \underline{\lambda})v_{xx}(u_2y, y) &= v_{yx}(u_2y, y), \\ (1 + \bar{\lambda})v_{xy}(u_1y, y) &= v_{yy}(u_1y, y), & (1 - \underline{\lambda})v_{xy}(u_2y, y) &= v_{yy}(u_2y, y), \end{aligned}$$

for $y > 0$.

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